Vacuum charge fractionization re-examined

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Abstract. We consider a model of a quantized fermion field that is based on the Dirac equation in one dimensional space and re-examine how the fermion number of the vacuum, or the vacuum charge, varies when an external potential is switched on. With this model, fractionization of the vacuum charge has been illustrated in the literature by showing that the external potential can change the vacuum charge from zero to a fractional number. Charge conservation then appears violated in this process. This is because the charge that has been examined in this context is only a part of the total charge of the vacuum. The total charge is conserved. It is not fractionalized unless the Dirac equation has a zero mode. Two other confusing aspects are discussed. One is concerned with the usage of the continuum limit and the other with the regularization of the current operator. Implications of these aspects of the vacuum problem are explored.

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1. introduction

The vacuum charge is the fermion number of the vacuum in units of the charge of the particle that is associated with the field under consideration. Throughout this paper we use the words "fermion number" and "charge" interchangeably. Jackiw and Rebbi [1] pointed out that, in the presence of a zero mode (a normalizable eigenstate of the Dirac equation with energy zero), the vacuum charge of the associated quantum field can take half-integral values. Their observation was followed by a large number of papers, mostly in the late 1970s and 1980s, in which various such possibilities were explored [2]. According to those analyses there are situations such that the vacuum charge can take not only half-integral but any fractional, irrational values. The notion of fractionization of the vacuum charge has been discussed in many different areas of physics.

The fractional charges that have been examined so far can be classified into two types, A and B, depending on how the charge varies when the external potential (or background field) involved is continuously varied. Type A is such that the charge remains invariant under the the variation of the external potential. The half-integral charge that was found by Jackiw and Rebbi [1] is of this type. There are other examples of this type in which the charge can take values $\pm\frac{1}{3}$ and $\pm\frac{2}{3}$ in certain units [3]. In

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type B, the charge (as it appeared in the literature) varies continuously as the external potential is varied. In type A it is essential that the external potential has a certain topological structure in terms of its large distance behavior. We do not question type A which we think is well understood. Rather we focus on type B. In type B it may appear as if some topology of the external potential is involved. As we will show in due course, however, the topology related to type B is actually a trivial one.

In this paper we consider a fermion field which is based on the Dirac equation in one-dimensional space and re-examine the charge of its vacuum. This model is a typical one with which fractionization of the vacuum charge has been illustrated in the literature [2]. We assume that there is no zero mode so that the charge fractionization of type A is absent. In the fractionization of type B, when an external potential is switched on, the vacuum charge changes from zero to a fractional value. It then appears that the charge is not conserved in this process. This is because, as we will emphasize, the charge that has been examined in this context in the literature is only a part of the total charge. Despite the long history of the vacuum problem, its aspect regarding fractionization versus conservation of the vacuum charge has not been fully addressed. The purpose of this paper is to clarify and explore implications of this and related aspects.

Throughout this paper we follow Dirac's hole theory. Alternatively one can use the second-quantized quantum field theory. We, however, find the hole theory more convenient in analysing the structure of the vacuum. The calculation of the vacuum charge involves the difference between two divergent quantities. The vacuum charge may or may not appear fractional depending on how the difference is calculated. We propose a natural way of resolving this ambiguity. This enables us to clearly identify the part of the charge that has been left out in earlier analyses. When the external potential is switched on the total vacuum charge remains conserved. It is not fractionalized. There are two other related aspects that we attempt to clarify. One is concerned with the usage of the continuum limit and the other with the regularization of the current operator.

Before examining the relativistic field with the Dirac equation it would be useful to review the time-honored Friedel sum rule for a nonrelativistic Fermi gas with an impurity placed in it [4]. This we do in Sec. II with emphasis on charge conservation. The impurity produces a potential, $V(r)$, that acts on the particles in the gas. If we choose $V(r)$ arbitrarily, it appears as if the charge induced around the impurity can take any fractional value. This is related to the usage of the continuum-energy limit. The total charge of the gas is conserved.

In Sec. III we examine a model of the Dirac field in one dimension. We see a situation very similar to the one found in Sec. II. In Sec. IV we examine the vacuum confined in a one-dimensional "bag" of a finite radius. Even such a finite bag is beset with an ambiguity that is related to the regularization of the current operator. A summary and discussions are given in Sec. V. We suggest an implication regarding the charge renormalization in quantum electrodynamics. We use units such that $c = 1$ and $\hbar = 1$ throughout this paper.

2. Friedel sum rule: Nonrelativistic Fermi gas with an impurity

Consider a nonrelativistic Fermi gas confined in a sphere of a large radius R (in three dimensions). We eventually let $R \to \infty$. When an impurity is placed in the gas, it induces a change in the charge density distribution. The free gas (without the impurity) is described by the Schrödinger equation

$$(H_0 - E)\phi(E, \mathbf{r}) = 0, \quad H_0 = \mathbf{p}^2/(2m), \tag{2.1}$$

In the presence of the impurity at the origin we have

$$(H - E')\psi(E', \mathbf{r}) = 0, \quad H = H_0 + V(r), \tag{2.2}$$

where $V(r)$ is the perturbation potential due to the impurity. It is a central, localized potential. The $\phi(E, \mathbf{r})$ and $\psi(E', \mathbf{r})$ both vanish at the boundary of the sphere, i.e., $r = R$, and are normalized within the sphere.

The energy levels are discrete. The E and E' are different. If it is attractive (repulsive), $V(r)$ shifts the energy levels downward (upward). There is a one-to-one correspondence between the perturbed and unperturbed energy levels. Imagine that we introduce the perturbation as $\eta V(r)$ and let η increase gradually from 0 to 1. Then the n-th E-level adiabatically goes to the n-th E'-level. We take this one-to-one correspondence for granted.

The density of the gas, defined as the deviation from that of the free gas, is given by

$$\rho(r) = \sum_{E' < E'_F} |\psi(E', \mathbf{r})|^2 - \sum_{E < E_F} |\phi(E, \mathbf{r})|^2, \tag{2.3}$$

where E_F (E'_F) is the free (shifted) Fermi energy. The total charge, which we define as the deviation from that of the free gas, is

$$Q = 4\pi \int_0^R r^2 \rho(r) dr = \sum_{E' < E'_F} - \sum_{E < E_F} . \tag{2.4}$$

When the perturbation is switched on, the total number of the particles in the gas remains the same and hence $Q = 0$. (See, however, the remarks given towards the end of this section.) We divide each of the density and the charge into two parts,

$$\rho(r) = \rho_c(r) + \rho_d(r), \quad Q = Q_c + Q_d, \tag{2.5}$$

$$\rho_c(r) = \sum_{E' < E_F} |\psi(E', \mathbf{r})|^2 - \sum_{E < E_F} |\phi(E, \mathbf{r})|^2, \tag{2.6}$$

$$\rho_d(r) = \mathrm{sign}(E'_F - E_F) \sum_{\{E_F, E'_F\}} |\psi(E', \mathbf{r})|^2, \tag{2.7}$$

$$Q_c = 4\pi \int_0^R r^2 \rho_c(r) dr = \sum_{E' < E_F} - \sum_{E < E_F} , \tag{2.8}$$

$$Q_d = 4\pi \int_0^R r^2 \rho_d(r) dr = \mathrm{sign}(E'_F - E_F) \sum_{\{E_F, E'_F\}} . \tag{2.9}$$

The radius R is finite, although very large. The energy levels are discrete. The $\sum_E$, which is the number of energy levels within the specified range, is well-defined and is an integer. Hence the Q's defined above are all integers.

Let the radial part of the normalized partial-wave function in the presence of the impurity be $\psi_l(E', r)$. In the asymptotic region where $V(r)$ is negligible, we obtain

$$\psi_l(E', r) = \left(\frac{1}{2\pi R}\right)^{1/2} \frac{1}{r} \sin\left[k'r + \eta_l(E') - \frac{l\pi}{2}\right], \tag{2.10}$$

where $k'^2/(2m) = E'$. In Eq. (2.6), the difference between $|\psi(E', \mathbf{r})|^2$ and $|\phi(E, \mathbf{r})|^2$ is small at large distances. Hence the distribution $\rho_c(r)$ is mostly concentrated around the impurity. By using the Schrödinger equation and Eq. (2.10) and by taking the continuum limit, that is, by replacing $\sum_E$ with $2(2R/\pi)\sum_l(2l+1)\int dk$, one can derive the well-known formula [4],

$$Q_c = \frac{2}{\pi}\sum_l (2l+1)\eta_l(E_F). \tag{2.11}$$

In deriving Eq. (2.11) one does not have to know the wave function in the vicinity of the impurity. If we choose $V(r)$ arbitrarily, the above Q_c can take any fractional value. This contradicts what we stated at the end of the preceding paragraph. We will discuss this contradiction shortly. For a real electron gas in a metal, only if $V(r)$ is chosen judiciously, Friedel's self-consistency condition or the Friedel sum rule

$$Q_c = Z \tag{2.12}$$

is satisfied. Here Z is the (excess) charge of the impurity and is an integer.

Next let us turn to $\rho_d(r)$. The Fermi momenta k_F of the free gas and k'_F of the perturbed gas are related by $k'_F R + \eta(E'_F) = k_F R$ and

$$k_F = k'_F - k_F = -\eta_l(E'_F)/R. \tag{2.13}$$

Unlike k_F, k'_F depends on l. If the impurity is introduced adiabatically, the particle population in each of the partial waves remains the same. The difference $E_F - E'_F$ is very small (because R is very large). If we replace $\sum_E$ of Eq. (2.7) with $2(2R/\pi)\Delta k_F \sum_l(2l+1)\int \delta(k-k_F)dk$, we obtain,

$$\rho_d(r) = -\frac{2}{\pi}\sum_l (2l+1)\eta_l(E_F){\psi_l}^2(E_F, r), \tag{2.14}$$

$$Q_d = -\frac{2}{\pi}\sum_l (2l+1)\eta_l(E_F). \tag{2.15}$$

This confirms $Q = Q_c + Q_d = 0$. To conserve the total charge of the gas, it is crucial to include $\rho_d(r)$ which stems from the change in the Fermi energy. In the asymptotic region where $V(r)$ is negligible, we obtain

$$\begin{aligned} 4\pi r^2 \rho_d(r) = & -\frac{4}{\pi R}\sum_l (2l+1)\eta_l(E_F) \\ & \times \sin^2\left[k_F r + \eta_l(E_F) - \frac{l\pi}{2}\right]. \end{aligned} \tag{2.16}$$

This distribution is diffuse; it spreads more or less uniformly over the entire asymptotic region. As $R \to \infty$ the density $\rho_{\rm d}(r)$ vanishes but Q_d does not.

We noted below Eq. (2.9) that Q_c and Q_d are integers. This is in contradiction with Eqs. (2.11) and (2.15) which show that Q_c and Q_d can be nonintegers and their dependence on $V(r)$ is continuous and smooth. The origin of this contradiction can be traced as follows. In deriving Eqs. (2.11) and (2.14), $\sum_E$ is replaced with $(R/\pi)\int dk$. This effectively smears out the discrete energies. The number of such smeared-out energies in a specified energy range becomes fuzzy. In Eq. (2.9) we count the number of energy levels in interval $\Delta E_F = E'_F - E_F$, which is of the same order of magnitude as the spacing of the discrete energy levels. In terms of k, the level spacing is $\Delta k = \pi/R$ which is comparable with $\Delta k_F = -\eta/R$ of Eq. (2.13). In such a case, smoothing the energy levels results in a considerable uncertainty. For the total charge Q, we count the number of states in interval $\{0, E_F\}$ which is much wider than the energy level spacing. This is why the smoothing does not affect Q $(= Q_c + Q_d)$ but it obscures each of Q_c and Q_d. The fractional values that appear through Eqs. (2.11) and (2.15) are artificial.

The model that we have examined above is just a mathematical model. Let us examine its relevance to a real electron gas in a metal. For example, the impurity can be a Zn ion of charge +2 which replaces a Cu ion of charge +1 in a metal lattice. The excess charge is $Z = 1$. In a real electron gas we have to remember the electron-electron interaction which is repulsive. This pushes the electrons in the diffuse distribution towards the surface of the metal. No diffuse distribution remains inside the metal. (This is a well-known result of electrostatics.)

We have assumed that the impurity potential $V(r)$ is switched on adiabatically. This, however, is not what actually takes place in the real electron gas. As pointed out by Friedel, when one of the ions is replaced by an impurity of excess charge Z, additional electron(s) are added to the system at the same time. The charge of the added electrons exactly cancels Z and the entire system, including its background ions, remains neutral [4, 5]. The Fermi energy of the gas with the added electron(s) becomes the same as the free Fermi energy E_F. If the electron-electron and electron-ion interactions are taken into account in a self-consistent manner, e.g., by means of the Hartree-Fock method, $V(r)$ will adjust itself such that $Q_c = Z$ is satisfied and the impurity charge is screened. Since Z is an integer, no fractional charge appears.

We raised a question regarding the continuum limit of the energy levels. We, however, do not propose to dismiss the Q_c and Q_d obtained in the continuum limit. It is conceivable that Eqs. (2.11) and (2.15) are practically acceptable or even preferable. The reason is as follows. Imagine that $V(r)$ is varied continuously and one of the E' levels crosses E_F. Then Q_c of Eq. (2.6) suddenly changes by $2(2l+1)$ where l is the angular momentum of the E' level. This is probably unrealistic. The energy levels near the Fermi energy may be only partially occupied. The fuzziness that is introduced by smearing the energies may be appropriate in such a situation. It is well-known that the Friedel sum rule (2.12) is useful [4, 5].

3. Dirac field in one dimension

We examine the vacuum charge of a Dirac field in one dimension. We start with the Dirac equation for a particle in a given external potential in single particle quantum mechanics. We compare the free and perturbed systems. The Dirac equation for the free system is

$$(H_0 - E)\phi(E, x) = 0, \quad H_0 = \alpha p + \beta m, \tag{3.1}$$

where $p = -id/dx$, α and β are the 2×2 Dirac matrices and m is the mass of the particle. In explicit calculations we use $\alpha = \sigma_2$ and $\beta = \sigma_3$. The Dirac equation for the perturbed system is

$$\begin{aligned} &(H - E')\psi(E', x) = 0, \\ &H = H_0 + U(x), \quad U(x) = \beta S(x) + V(x), \end{aligned} \tag{3.2}$$

where $S(x)$ is a Lorentz scalar and $V(x)$ the zero-th component of a Lorentz vector.

If $V(x) = 0$, there is a symmetry between the positive and negative energy spectra: $\psi(E', x)$ and $\psi(-E', x)$ are related by $\psi(-E', x) = \alpha\beta\psi(E', x)$. Because there cannot be energy-degeneracy in one dimension, this symmetry may seem to exclude the possibility of a zero mode. A zero mode, however, is allowed provided that $S(x)$ has a topology such that $m + S(\infty)$ and $m + S(-\infty)$ are of opposite signs [1, 6]. Such a situation arises with a kink soliton background. The zero mode remains no matter how $S(x)$ is modified as long as this topology is maintained. If $V(x) \neq 0$, the symmetry between the positive and negative energy spectra is broken. But the zero mode can still exist as long as $[m + S(\infty)][m + S(-\infty)] < 0$ and $|V(x)| < |S(x)|$ when $x \to \pm\infty$. The zero mode is stable against the variation of $V(r)$. This is the type of zero mode that underlies half-integral charges [1].

In this paper we assume that $S(x)$ has no such topology as described above. To be more explicit, we assume that $[m + S(\infty)][m + S(-\infty)] \geq 0$. Under this assumption for $S(x)$, it is still possible to have a zero mode. This can be done by carefully adjusting $V(x)$ such that one of the eigenvalues becomes zero. This zero mode is obviously unstable against the variation of $V(x)$. Its energy eigenvalue can deviate from zero even if the variation of $V(x)$ is very slight. We may say that the existence of this zero mode is accidental. This is in contrast to the zero mode that we described in the preceding paragraph. In any case we assume that our Dirac Hamiltonian H has no zero mode.

The H of Eq. (3.2) can be transformed into different forms. An example is,

$$\begin{aligned} H_f &\equiv e^{i\alpha f(x)} H e^{-i\alpha f(x)} \\ &= \alpha p + V(x) - \frac{df(x)}{dx} \\ &+ \beta[m + S(x)][\cos 2f(x) - i\alpha \sin 2f(x)]. \end{aligned} \tag{3.3}$$

The term with $-i\beta\alpha = -i\sigma_3\sigma_2 = -\sigma_1$ is a pseudoscalar potential. Potential $V(x)$ can be eliminated by choosing $f(x)$ such that $df(x)/dx = V(x)$. If $m + S(x) = 0$, then the transformed Hamiltonian H_f simply becomes $H_0 = \alpha p$, which is of course solvable. We

will make use of this transformation in the next section. In this section we assume that the mass is nonzero.

Suppose we eliminate $V(x)$ in H_f by means of $df(x)/dx = V(x)$. It is understood that $m + S(\infty)$ and $m + S(-\infty)$ are of the same sign. Assume that $f(\infty)f(-\infty) < 0$. Then the peudoscalar part of the potential of H_f, which is proportional to $i\alpha\beta \sin 2f(x)$, has a topology, that is, its asymptotic values for $x \to \pm\infty$ are of opposite signs. If we go back to H, however, there is no such topology because $df(x)/dx$ has the same sign at $x \to \pm\infty$. Hamiltonians of the form of H_f are extensively used in the literature [2] and one might have the impression that the fractional charge associated with such H_f is due to its topology. This topology, however, is a trivial one in the sense that it can be transformed away. Let us emphasize that the topology carried by the Lorentz scalar potential remains the same in transformation (3.3). The topology of $S(x)$ that we mentioned in the paragraph below Eq. (3.2) cannot be transformed away.

We return to Hamiltonian H of Eq. (3.2). We assume that the system is confined within a box of large radius R, i.e., $r = |x| < R$. We do this by assuming an infinite square-well potential of the Lorentz scalar type, which is ∞ for $r > R$. It is understood that this square-well potential is a part of $S(x)$. The wave function has to satisfy a boundary condition at $r = R$; see Eq. (4.1) of the next section. Since R is finite, the energy levels are discrete. We let $R \to \infty$ in the end. (We keep R finite in Sec. IV.) The wave function vanishes for $r > R$. Therefore, $S(x)$ and $V(x)$ for $r > R$ do not appear in the following calculations. We assume that $S(x)$ and $V(x)$ within the box are both localized around the origin. In this way we can see a parallelism between this and the preceding sections more easily. In the following calculations, however, the $S(x)$ and $V(x)$ within the box can be chosen arbitrarily.

We can find a one-to-one correspondence between the free energy levels and the perturbed ones as we did in Sec. II. For simplicity we assume that positive Es correspond to positive E's. In other words, the perturbation does not change the sign of the energy. The vacuum is such that the negative energy levels are all occupied. The number of the negative energy levels is of course infinite. We, however, first assume that the negative energy levels are filled only down to a certain Fermi energy $E_F < 0$ for the free vacuum and similarly to $E'_F < 0$ for the perturbed vacuum. Eventually we let the Fermi energies tend to $-\infty$. Unlike the Fermi energy of the nonrelativistic Fermi gas, the Fermi energy of the vacuum is a mathematical device. We need such a device so that we can clearly keep track of the number of particles in the vacuum.

The density and the charge of the perturbed vacuum are respectively defined as the deviations from those of the free vacuum,

$$\rho(x) = \sum_{0>E'>E'_F} |\psi(E', x)|^2 - \sum_{0>E>E_F} |\phi(E, x)|^2, \tag{3.4}$$

$$Q = \int_{-R}^{R} \rho(x)dx = \sum_{0>E'>E'_F} - \sum_{0>E>E_F} . \tag{3.5}$$

The one-to-one correspondence between the free and perturbed energy levels implies $Q = 0$. When the perturbation is switched on, the total charge remains conserved.

Exactly in the same way as we did in Sec. II we divide each of the density and the charge into two parts,

$$\rho(x) = \rho_c(x) + \rho_d(x), \quad Q = Q_c + Q_d, \tag{3.6}$$

$$\rho_c(r) = \sum_{0>E'>E_F} |\psi(E', x)|^2 - \sum_{0>E>E_F} |\phi(E, x)|^2, \tag{3.7}$$

$$\rho_d(r) = \mathrm{sign}(E_F - E'_F) \sum_{\{E_F, E'_F\}} |\psi(E', x)|^2, \tag{3.8}$$

$$Q_c = \int_{-R}^{R} \rho_c(x) dx = \sum_{0>E'>E_F} - \sum_{0>E>E_F}, \tag{3.9}$$

$$Q_d = \int_{-R}^{R} \rho_d(x) dx = \mathrm{sign}(E_F - E'_F) \sum_{\{E_F, E'_F\}}. \tag{3.10}$$

The Q can be rewritten as

$$Q = \frac{1}{2} \left(\sum_{0>E'>E'_F} - \sum_{0<E'<E''_F} \right). \tag{3.11}$$

Imagine the positive-energy level of $E = |E_F|$ of the free system. The $E''_F > 0$ of Eq. (3.11) is the energy level (of the perturbed system) that corresponds to $|E_F|$. Figure 1 schematically shows how the Fermi energy shifts when the perturbation is attractive. In general $E''_F \neq -E'_F$. In deriving Eq. (3.11) we used

$$\sum_{0>E'>E'_F} + \sum_{0<E'<E''_F} = 2 \sum_{0>E>E_F}, \tag{3.12}$$

which is based on the one-to-one correspondence between Es and E's. Furthermore the Q_c can be written as

$$Q_c = \frac{1}{2} \left(\sum_{0>E'>E_F} - \sum_{0<E'<|E_F|} \right). \tag{3.13}$$

The Q_c is a measure of the asymmetry between the positive and negative energy spectra.

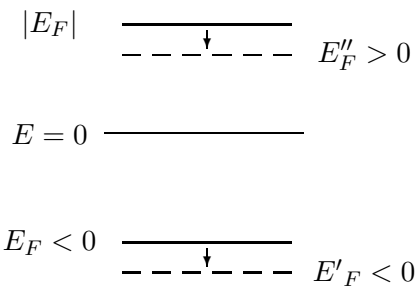

Fig. 1 Unperturbed and perturbed Fermi energies.

When E_F, E'_F and R are finite, the $\sum_E$'s are all well-defined. The above Q's are all integers. When $E_F \to -\infty$, $E'_F \to -\infty$ and $R \to \infty$, the Q's remain as the same integers. In Sec. II and so far in this section we have distinguished the unperturbed and perturbed energies, using different notations E and E'. In the following, when the distinction between E and E' is unimportant or is clear from the context, we may denote both of them as E for notational brevity.

The $\rho_c(x)$ is the one that has been extensively examined in the literature, for example, in Refs. [7, 8]. According to those analyses it is given by

$$Q_c = -\frac{1}{\pi}\eta(-\infty), \tag{3.14}$$

where we have taken the limit of $E_F \to -\infty$. The phase shift $\eta(E)$ pertains to the transmission coefficient $T(E)$. In the Born approximation the phase shifts are given by

$$\eta(E) = -\frac{1}{k}\int_{-R}^{R}[EV(x) + mS(x)]dx, \tag{3.15}$$

which becomes exact when $|E| \to \infty$. Thus we arrive at

$$Q_c = -\frac{1}{\pi}\int_{-R}^{R} V(x)dx. \tag{3.16}$$

The $S(x)$ has no effect on Q_c. It follows from Eq. (3.15) that $\eta(\infty) = -\eta(-\infty)$ and Eq. (3.14) can be written as $Q_c = (1/2\pi)[\eta(\infty) - \eta(-\infty)]$. This shows that Q_c is a measure of the asymmetry between the positive and negative energy phase shifts. Let us add that, in the nonrelativistic version of the present model, the phase shifts in the Born approximation are $\eta(E) = -(m/k)\int_{-R}^{R} V(x)dx$. Unlike its relativistic counterpart, the nonrelativistic $\eta(E)$ vanishes as $E \to \infty$.

No attention seems to have been paid to $\rho_d(r)$ in the literature so far. In order to see similarity to the calculation of Sec. II clearly, let us assume that $U(x)$ of Eq. (3.2) is an even function of x so that parity is a good quantum number. There are two partial waves, one with even parity and the other with odd parity [9]. The transmission coefficient $T(E)$ is related to the partial wave phase shifts $\eta_\pm$ by $T = [e^{2i\eta_+} + e^{2i\eta_-}]/2$ and hence

$$\eta(E) = \eta_+(E) + \eta_-(E). \tag{3.17}$$

In the asymptotic region where $U(r)$ is negligible, the perturbed wave function of energy $|E| > m$ and positive parity takes the following form,

$$\psi_+(E,x) = \sqrt{\frac{E+m}{2RE}}\begin{pmatrix} \cos(kr+\eta_+) \\ \frac{k}{E+m}\hat{x}\sin(kr+\eta_+) \end{pmatrix}, \tag{3.18}$$

where $k = \sqrt{E^2 - m^2}$ and $\hat{x} = x/r$. Suffix + of the phase shift $\eta_+(E)$ refers to positive parity. There may be bound states with $|E| < m$. For negative parity we obtain

$$\psi_-(E,x) = \sqrt{\frac{E-m}{2RE}}\begin{pmatrix} \frac{k}{E-m}\hat{x}\sin(kr+\eta_-) \\ \cos(kr+\eta_-) \end{pmatrix}. \tag{3.19}$$

The $\phi_\pm(E,x)$ are obtained by dropping $\eta_\pm$.

The Fermi momenta before and after perturbation $U(x)$ is switched on, k_F and k'_F, are related again by Eq. (2.13). By taking the continuum limit, we obtain

$$\rho_d(x) = \frac{1}{\pi}\sum_{l=\pm}\eta_l(E_F)|\psi_l(E_F,x)|^2. \tag{3.20}$$

By integrating this with respect to x and letting $E_F \to -\infty$ we obtain

$$Q_d = \frac{1}{\pi}\eta(-\infty) = \frac{1}{\pi}\int_{-R}^{R} V(x)dx. \tag{3.21}$$

In the asymptotic region where $U(x)$ is negligible, in the limit of $E_F \to -\infty$, we find that $|\psi_+(E,x)|^2 \to 1/(2R)$ and hence $\rho_d(x) \to (1/\pi R)\eta(-\infty)$.

The remark given in Sec. II regarding Q_c and Q_d also applies to the Q_c and Q_d obtained above. As a model, $V(x)$ can be chosen arbitrarily. Then Q_c of Eq. (3.16) and Q_d of Eq. (3.21) can take any fractional values. As we noted below Eq. (3.13), however, the Q's of Eqs. (3.9) and (3.10) are integers. The fractional values of Q_c of Eq. (3.16) and Q_d of Eq. (3.21) are mathematical artifacts that stem from replacing discrete summation $\sum_E$ with the integration $(R/\pi)\int dk$ over the smoothed energy levels.

At the end of Sec. II we suggested a practical justification for Q_c of Eq. (2.11) of the nonrelativistic electron gas in a metal. Such justification is untenable for the present relativistic model in which the Fermi energy is a purely mathematical device. Towards the end of this section we argue that, although they should not be taken literally, these artificial results do no harm in real physics.

Let us briefly discuss an explicit example. Assume that

$$S(x) = 0, \quad V(x) = -2\lambda\delta(x). \tag{3.22}$$

This is a special case ($\lambda_1 = \lambda_2 = \lambda$) of the model that was examined in Ref. [10]. This model is related to the model examined by MacKenzie and Wilczek [7] by transformation (3.3). In dealing with the δ-function potential, we start with a square-well potential and after solving the Dirac equation we take the narrow-width limit of the square well-potential. In this connection see Ref. [11]. The phase shifts are given by

$$\tan\eta_\pm(E) = \frac{E \pm m}{k}\tan\lambda. \tag{3.23}$$

If $g < 0$ there is a bound state of positive parity. Its energy is

$$E = m\cos(2\lambda). \tag{3.24}$$

The Dirac matrices used in Ref. [10] are $\alpha = \sigma_1$ and $\beta = \sigma_3$, which can be transformed to $\alpha = \sigma_2$ and $\beta = \sigma_3$ that we are using by the unitary transformation $\exp(i\pi\sigma_3/4)$. The wave function of the bound state is

$$\psi(x) = \sqrt{\frac{\kappa(E+m)}{2m}}\begin{pmatrix} 1 \\ \frac{\kappa}{E+m}\hat{x} \end{pmatrix} e^{-\kappa r}, \tag{3.25}$$

where $\kappa = \sqrt{m^2 - E^2}$. If $g > 0$ there is a bound state of negative parity but we do not delve into such details in this paper. The $\rho_c(x)$ was worked out explicitly in Ref. [10]. The summation $\sum_E$ was done as $(R/\pi)\int dk$. The $\rho_c(x)$ is indeed concentrated around

the origin. It decays like e^{-2mr}. If we let $m \to 0$, $\rho_c(x)$ becomes uniform. By carrying out the integration $\int \rho_c(x)dx$ explicitly, one obtains Q_c of Eq. (3.16). The $\rho_d(x)$ was not considered in Ref. [10] but it is given by Eq. (3.20). It is uniform. Let us add that a model in which the δ-function potential is replaced by a square-well potential has been also examined [12]. The square well-version is related to the model of Ref. [13] by transformation (3.3). This ends the discussion of the example.

As we pointed out already the fractional charges of Eqs. (3.16) and (3.21) are mathematical artifacts. Still let us speculate on what happens if we accept the fractional charges literally. In the model that we have examined, the particles individually interact with the external potential but there is no interaction between the particles. We assumed that the external potential can be chosen arbitrarily. In real physical systems, however, the particles generally interact with each other. When the inter-particle interaction is (approximately) eliminated in a self-consistent manner, the potential that emerges is not an arbitrary one. If the particle of the field is the electron in a metal, the remarks given at the end of Sec. II apply. The repulsion between electrons pushes the electrons in the diffuse distribution towards the edge at $r = R$. Such a charge accumulated at the edge can easily move on to another object that comes into contact with the metal. For Q_c, the Friedel sum rule (2.12) will be satisfied. Thus the self-consistency prevents fractionization of the vacuum charge.

Our model can be regarded as a one-dimensional simulation of the relativistic nuclear shell model. In that case we have a number of nucleons in the positive energy levels, but let us focus on the vacuum effect. According to the Dirac phenomenology of nuclear physics, the relativistic potential of the nuclear shell model is of the form of $U(r) = \beta S(r) + V(r)$ [14]. For medium to heavy nuclei, $S(r)$ and $V(r)$ are well represented by the Wood-Saxon form. Inside the nucleus $S + V \approx -50$ MeV and $-S + V \approx 800$ MeV. The potential acting on the lower component of the Dirac wave function is $-S + V$, which is almost as large as the nucleon rest mass of 940 MeV. Consequently relativistic effects can be appreciable even at very low energies. If we assume $V = 400$ MeV, we obtain $Q_c = 400(2R_0/\pi)$ MeV·fm where $R_0 \approx (1.2)A^{1/3}$ is the radius in fm of the nucleus of mass number A. If we assume $A = 100$, for example, we find $R_0 \approx 5.6$ fm and $Q_c = -Q_d \approx -7.2$. Approximately 7 nucleons are moved from the concentrated region to the diffuse distribution. This number is not negligibly small.

In our model calculation the diffuse distribution is uniform in the asymptotic region (outside the range of the shell model potential). Unlike the electron case, the interaction between the nucleons is attractive. When the nucleons are treated in a self-consistent manner, like the Hartree-Fock method, the interaction between the nucleons will produce an attractive potential and the diffuse distribution will be pulled back to a finite region. It may become something like a nuclear halo. The nucleons, including those in the diffuse distribution, will form a nucleus of a finite size. The total charge of the nucleus is an integer. It is an interesting challenge to do such a self-consistent calculation including the vacuum effect. If the nucleons in the diffuse distribution somehow escaped to infinity, the nucleus would be left with Q_c. If this Q_c takes a fractional value, the nuclear charge

becomes fractional. This would lead to bizarre consequences. We know that this is not the case in reality.

Finally let us mention super-heavy quasimolecules that could be formed in a collision of two very heavy ions. This subject has been extensively discussed by Greiner et al. [15]. When the strength of the Coulomb potential due to the merging nuclei exceeds a certain critical value, the lowest positive energy level of the electron dives into the negative energy sea. This may lead to the creation of an electron-positron pair. Suppose that the vacuum charge is fractionalized in this process. Then can a fraction of the pair be created? This does not seem to make sense.

4. Vacuum confined in a finite bag

We examine a one-dimensional version of the so-called MIT bag model [16]. We also discuss the chiral bag model [17] towards the end of this section. We keep the radius R of the bag finite. In Sec. III we divided the density into two parts, concentrated and diffuse. Such division is not very meaningful when R is not very large. In Secs. II and III, we pointed out that replacing $\sum_E$ with $(R/\pi)\int dk$ leads to fractional charges. When R is finite and not very large, one may think that there is no room for such complication and no problem arises regarding charge conservation because the charge cannot escape to infinity. Nevertheless, the vacuum charge may be fractionalized in the bag, depending on how it is calculated.

The Hamiltonian of the bag model is the same as that of Sec. III. We assume an infinite square well potential for $r > R$ so that the particle is confined to $r < R$. This leads to the boundary condition at $r = R$,

$$(1 - i\alpha\beta\hat{x})\psi(x) = (1 + \sigma_1\hat{x})\psi(x) = 0, \tag{4.1}$$

which is equivalent to Eq. (14) of Ref. [18]. This condition makes the scalar density $\psi^\dagger\beta\psi$ vanish at $r = R$.

We begin with the simple case of $m = 0$. The solutions of the free Dirac equation can be classified in terms of parity. For the unperturbed system with H_0, we obtain

$$\phi_+(x) = N\begin{pmatrix} \cos kx \\ -\sin kx \end{pmatrix}, \tag{4.2}$$

$$\phi_-(x) = N\begin{pmatrix} \sin kx \\ \cos kx \end{pmatrix} = -i\alpha\phi_+(x), \tag{4.3}$$

where suffix $\pm$ refers to parity and $N = 1/\sqrt{2R}$ is the normalization factor. The density $|\phi(E,x)|^2 = 1/(2R)$ is a constant. The k and $E = \pm k$ are determined by the boundary condition (4.1). The positive energy eigenvalues are given by

$$E_n{}^{(\pm)} = \pm\frac{(2n-1)\pi}{4R}, \quad n = 1, 2, 3, \dots, \tag{4.4}$$

where superscript $(\pm)$ refers to the sign of the energy. All energy levels are equally spaced with interval $\pi/(2R)$.

We now examine the perturbed system with the $H = H_0 + V(x)$. We assume $S(x) = 0$ for simplicity. The H and H_0 can be related by transformation (3.3). The following two wave functions satisfy the perturbed Dirac equation,

$$\psi_+(x) = N \begin{pmatrix} \cos[k'x - f(x)] \\ -\sin[k'x - f(x)] \end{pmatrix}, \quad \frac{df(x)}{dx} = V(x), \tag{4.5}$$

and $\psi_-(x) = -i\alpha\psi_+(x)$. In order for the above to satisfy the Dirac equation, $V(x)$ does not have to be an even function of x. If $V(x)$ is not an even function, parity is not a good quantum number. We still retain the suffixes $\pm$ because the above solutions are respectively related to the free solutions $\phi_\pm(x)$. The above is a generalization of the solution of Ref. [19] in which $V(x) = \lambda x$ was assumed.

The perturbed density is $|\psi(E', x)|^2 = 1/(2R)$. Imposing the condition (4.1) we find that the perturbed energies and the energy shift,

$$\Delta E = E' - E = \frac{1}{2R}\int_{-R}^{R} V(x)dx. \tag{4.6}$$

This ΔE has two remarkable aspects. (i) The ΔE is of the form of the first-order perturbation; yet it is exact. This is so no matter how strong $V(x)$ is. Higher order terms are absent. If $V(x)$ is an odd function of x, then $\Delta E = 0$. (ii) This energy shift ΔE applies to all energy levels, that is, all energy levels are shifted by exactly the same amount.

In the vacuum all negative energy levels are occupied. Start with the free vacuum and switch on perturbation $V(x)$. Assume that $V(x)$ is attractive so that $\Delta E < 0$. Recall that all energy levels are equally spaced with separation $\pi/(2R)$. If $|\Delta E| < \pi/(2R)$, the lowest positive energy remains positive and hence the number of the negative energy levels remains the same. There is no change in the vacuum charge, i.e., $Q = 0$. If $\pi/(2R) < |\Delta E| < \pi/R$, the lowest positive E becomes negative E'. Therefore the number of the negative energy levels increases by one. This new negative energy level is empty. If we fill this level by hand, the charge of the new vacuum becomes one unit greater than that of the free vacuum. No fractional charge appears.

Suppose that $|\Delta E| \gg \pi/(2R)$. Then a large number of positive energy levels move into negative energy sea. The number of such levels can be estimated as

$$-\frac{\Delta E}{\pi/(2R)} = -\frac{1}{\pi}\int_{-R}^{R} V(x)dx. \tag{4.7}$$

We have put the negative sign in the left hand side of Eq. (4.7) because the number should be positive when ΔE is negative. It is interesting that the above result is consistent with the Q_c of Eq. (3.16). However, the number of the levels that goes from positive to negative should always be an integer. Equation (4.7) is only a fuzzy estimate of the discrete number.

Equation (3.11) for Q also holds in the present case. When $E'_F \to -\infty$ and $E''_F \to \infty$, Q is the difference of two divergent series. The difference depends on how the summation is done. We propose to do this in the same way as we did in Sec. III. We first keep E'_F and E''_F finite. Then the one-to-one correspondence between the Es

and E's dictates that $Q = 0$. This remains so in the limit of $E'_F \to -\infty$ and $E''_F \to \infty$. This we believe is a most natural prescription for dealing with the series.

There is another prescription that has been widely used [20]. This is to rewrite Q as

$$Q = \frac{1}{2} \lim_{s \to +0} \left(\sum_{0>E'} e^{-sE'} - \sum_{0<E'} e^{sE'} \right), \tag{4.8}$$

where the summation is extended to infinity. (No Fermi energy is assumed.) This prescription is based on the so-called split-point regularization of the density operator in quantum field theory [21]. Let us apply this method to our bag model. With ${E'}_n^{(\pm)} = {E_n}^{(\pm)} + \Delta E$ where ${E_n}^{(\pm)}$ is that of Eq. (4.9) we obtain

$$\sum_{0>E'} e^{-sE'} = \frac{e^{s\Delta E} \exp\left(\frac{\pi s}{4R}\right)}{1 - \exp\left(-\frac{\pi s}{2R}\right)}, \tag{4.9}$$

and similarly for the other summation. By taking the limit of $s \to +0$ and using Eq. (4.6) we obtain

$$Q = -\frac{1}{\pi} \int_{-R}^{R} V(x) dx. \tag{4.10}$$

We have three remarks to make on the above result.

(i) Equation (4.10) follows no matter how small R is and hence no matter how large the level spacing is. If the level spacing is very large and $V(x)$ is not very strong, all positive energy levels remain positive. The number of the negative-energy levels does not change. Even then Eq. (4.10) implies that the vacuum charge changes, which we find unphysical. The current is zero everywhere in stationary states. The charge cannot escape from the bag and hence it must be conserved.

(ii) Imagine that the energy levels are continuous and replace the sum over $n = 1, 2, 3, \ldots$ of Eq. (4.8) with an integral $\int^{\infty} dn$. We still obtain Eq. (4.10). (This result is independent of the lower end of the integral.) In this sense the prescription (4.8) is equivalent to smearing out the discrete energy levels. It is interesting that this smearing leads to the same fractional charge as that of Q_c of Eq. (3.16) which is also related to the smearing of the energy levels as we discussed in detail in Sec. II. We think this is a mathematical artifact.

(iii) Divide Q into Q_c and Q_d as we did in Sec. III. (This division is not interesting in the sense that, when $m = 0$, ρ_c and ρ_d are both diffuse.) If we use Eq. (4.8), we find that Q_d disappears. The prescription (4.8) is equivalent to ignoring the change in the Fermi energy in our proposed scheme. In the case of Sec. III, Q_c and Q_d may both be fractionalized but they cancel each other, $Q = Q_c + Q_d = 0$. In the present case, fractionalized Q_c remains because Q_d is missing. The total charge is not conserved. This is unphysical.

So far we have assumed $m = 0$. We now assume that $m > 0$. The Dirac equation can be solved exactly for H_0 but not for H in general. As $|E'|$ becomes much larger than m, however, the effect of the finite mass becomes negligible. For such E's the

energy shift is given by Eq. (4.6). The one-to-one correspondence between the Es and E's implies $Q = 0$. Here we are again assuming that positive E's correspond to positive E's, for simplicity. No fractional vacuum charge appears.

Let us see what happens if we use prescription (4.8) when $m > 0$. The Q is essentially determined by the energy shift for very large E's. This situation is similar to the one we saw in Sec. III where Q was determined by $\eta(\pm\infty)$. It is not difficult to see by using Eq. (4.8) for the cases of $m > 0$ and $m = 0$ and remembering that the E' becomes independent of m as $|E'| \to \infty$, that

$$Q(m > 0) - Q(m = 0) = 0. \tag{4.11}$$

Hence Eq. (4.10) also holds even when $m > 0$.

We now turn to the chiral bag model [17, 20, 22]. Fractionization of the vacuum charge has been extensively discussed for the chiral bag model, but never for the MIT bag model that we are using. This may give the wrong impression that fractionization is a peculiar feature of the chiral bag model. As far as the vacuum charge associated with the bag is concerned, however, there is no essential difference between the two types of the bag models.

It is instructive to examine the one-dimensional version of the chiral bag model that was proposed by Zahed [22]. The quark is assumed to be massless. The quark wave function $q(x)$ for a stationary state obeys the Dirac equation

$$(\alpha p - E)q(x) = 0, \tag{4.12}$$

inside the bag and the boundary condition at $x = \pm R$

$$[\cos\theta(x) - i\alpha\sin\theta(x) + \sigma_3\hat{x}]q(x) = 0, \tag{4.13}$$

where $\alpha = \sigma_2$ and $\theta(x)$ is a function that pertains to the soliton with which the quark interacts. If we write $q(x)$ as

$$q(x) = \exp\{(i/4)[2\theta(x) - \pi]\alpha\}\psi(x), \tag{4.14}$$

Eq. (4.12) becomes

$$\left[\alpha p + \frac{1}{2}\frac{d\theta(x)}{dx} - E\right]\psi(x) = 0, \tag{4.15}$$

and the boundary condition (4.13) is reduced to Eq. (4.1). As far as the quarks in the bag are concerned, Zahed's model is exactly equivalent to the one-dimensional MIT bag model with external potential $V(x) = (1/2)d\theta(x)/dx$ and the quark mass $m = 0$. The energy spectrum and the energy shift that Zahed obtained are the same as those of of the MIT bag model. This situation also holds in three dimensions, that is, the chiral bag model can be transformed into the MIT bag model with an external potential.

From our point of view, there is no anomalous, fractional baryon number associated with the bag. We can still have a soliton which interacts with the bag, and the bag plus the soliton can be interpreted as a baryon. The baryon number carried by the bag, as we see it, is an integer. There is no need to invoke the interpretation that a part of the baryon number is carried by the soliton as is done in the chiral bag model [22, 23].

We should add that not all the people who worked with the chiral bag model assumed that the baryon number in the bag is fractional. In the models for baryons developed in Ref. [24], the baryon number of the chiral bag is unity. Examining the vacuum charge of a model similar to that of Zahed, Banerjee argued that "there is nothing wrong about departing from the split-point form" in regularizing the current operator [25]. We concur with him in this respect. Banerjee then suggested a regularization scheme such that, even in the presence of spectrum asymmetry that is induced by an external perturbation, one uses $e^{\pm sE}$ instead of $e^{\pm sE'}$ for Q of Eq. (4.8). This ad hoc scheme leads to $Q = 0$. We believe, however, the scheme with the Fermi energy for the vacuum that we have introduced in this paper is more natural than what Banerjee suggested.

5. Summary and Discussion

We have examined the vacuum charge for a fermion field model based on the one-dimensional Dirac equation with an external potential. This model is a typical one with which fractionization of the vacuum charge has been illustrated in the literature. We classified the fractional vacuum charges so far examined in the literature into two types, A and B. When the external potential of the model is continuously varied, the vacuum charge of type A remains unchanged, whereas that of type B varies continuously. We assumed that there is no zero mode for the Dirac equation so that the charge fractionization of type A does not appear. We have focussed on the charge fractionization of type B. We pointed out that the topology that has been mentioned in the literature in connection with type B is of a trivial nature.

The vacuum charge, which is the difference between the charge of the perturbed vacuum and that of the free vacuum, is in the form of the difference between two divergent series. The result depends on how the difference is calculated. We proposed a natural way of resolving this ambiguity. For the negative energy states of the free vacuum, we assume a Fermi energy $E_F < 0$. All the energy levels between 0 and E_F are occupied and those below E_F are empty. When the external potential is switched on the Fermi energy shifts, $E_F \to E'_F$ but the one-to-one correspondence between the Es and E's is maintained. We eventually let $E_F \to -\infty$ and $E'_F \to -\infty$. This device is instrumental in clearly keeping track of the number of the particles in the vacuum.

In Sec. III we examined a model that extends to the entire one-dimensional space. We divided the total charge of the vacuum Q into two parts, Q_c and Q_d. The Q_c is the part of the charge that has been examined extensively in the literature. In order to see the charge conservation $Q = Q_c + Q_d = 0$, it is crucial to include Q_d that is related to the change in the Fermi energy due to the external potential. Each of Q_c and Q_d may take a fractional value but we pointed out that this is a mathematical artifact that arises from replacing $\sum_E$ with $(R/\pi)\int dk$. In this continuum limit the discrete energy levels become effectively smeared out. We emphasized that the level spacing and the energy shift in this situation are of the same order of magnitude. The number of

smeared out energy levels in a specified range becomes fuzzy. This artifact, however, does not jeopardize the conservation of the total charge.

The model that we examined in Sec. III can be taken as a one-dimensional simulation of the relativistic shell model of the atomic nucleus. If we take the fractional charge literally, this leads to bizarre consequences as we discussed towards the end of Sec. III. If the shell model potential is constructed in a self-consistent manner, we expect that the diffuse distribution $\rho_d(x)$ becomes confined to a finite region around the nucleus. The total nuclear charge is not fractionalized.

In Sec. IV we examined the vacuum charge of a finite bag confined within $|x| < R$. Since R is finite and not very large, we do not replace $\sum_E$ with $\int dk$. Nevertheless a fractional charge can appear when we use the summation method, Eq. (4.8), that is related to the split-point regularization of the current operator. This occurs in the MIT bag model as well as in the chiral bag model. We concur with Banerjee [25] in arguing that this outcome of Eq. (4.8) is unnatural. From our point of view, no fractionization of the charge or the quark number appears in the bag models.

What we have discussed in this paper may have relevance to the charge renormalization of quantum electrodynamics (QED). Suppose a test charge e_0 is placed in the vacuum. The e_0 is renormalized to e,

$$e = e_0(Z_2/Z_1)\sqrt{Z_3}. \tag{5.1}$$

The Ward identity leads to $Z_1 = Z_2$. We are still left with Z_3 which is due to the vacuum polarization; see, e.g. Ref. [26]. If we calculate Z_3 by perturbation theory, it is represented by a divergent integral. If we cut-off the integral, Z_3 depends on the cut-off parameter. Depending on the choice of the cut-off parameter, Z_3 can take any fractional value. This was probably the first example in which a fractional charge appeared. The process that is responsible for the charge renormalization is typically through the electron-positron pair creation in the virtual state. Because the charge of this pair is zero, however, it cannot change the charge of the system unless part of the charge somehow disappears. The mechanism of $Z_3 \neq 1$ is not clear. If we consider the same problem but assuming that the space is confined in a finite cavity, the situation will be similar to the one that we discussed in Sec. IV. Whatever happens to the system, the total charge is confined within the cavity and hence it is conserved. This seems to mean $Z_3 = 1$. If we let the radius of the cavity tend to infinity, we will still have $Z_3 = 1$. Would it be possible to reformulate QED such that we do not have to consider the charge renormalization?

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